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# M2-branes on M-folds

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Abstract: We argue that the moduli space for the Bagger-Lambert $A_4$ theory at level $k$ is $(\mathbb{R}^8 \times \mathbb{R}^8)/D_{2k}$, where $D_{2k}$ is the dihedral group of order $4k$. We conjecture that the theory describes two M2-branes on a $\mathbb{Z}_{2k}$ "M-fold", in which a geometrical action of $\mathbb{Z}_{2k}$ is combined with an action on the branes. For $k = 1$, this arises as the strong coupling limit of two D2-branes on an $O2^-$ orientifold, whose worldvolume theory is the maximally supersymmetric SO(4) gauge theory. Finally, in an appropriate large-$k$ limit we show that one recovers compactified M-theory and the M2-branes reduce to D2-branes.

Keywords: D-branes, M-Theory.

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## Contents



## 1. Introduction

A new class of conformal invariant, maximally supersymmetric field theories in $2+1$ dimensions has been found recently [1, 2]. These theories are based on "3-algebras" and include a non-dynamical gauge field with a Chern-Simons-like interaction. They have several striking properties including the absence of continuous marginal deformations. The motivation for studying these theories was to find a Lagrangian description of the conformally invariant fixed point of maximally supersymmetric Yang-Mills theories in $2+1$ dimensions, which is believed to describe the worldvolume dynamics of coincident membranes in M-theory.

While the 3-algebra theories share many features with the expected M2-brane theories, they also give rise to some puzzles. One is that only a single 3-algebra, denoted $A_4$, is presently known, so an explicit theory exists for at best a small fixed number of membranes. It was proposed in ref. [3] that this number is 3, which suggests the surprising possibility that the IR theory on 2 D2-branes is trivial. Also somewhat puzzling was how parity could be preserved when the gauge field has Chern-Simons interactions.

Some of these puzzles have been resolved in recent days [4−6]. For the $A_4$ 3-algebra, all these papers (as well as ref. [7]) found that the theory could be recast as an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge theory. In ref. [4] it was further shown that giving one of the scalars a vev reduces the 3-algebra action to a strongly coupled supersymmetric SU(2) Yang-Mills action by a novel Higgs mechanism. This renders one combination of the two Chern-Simons fields massive and the other one dynamical in consequence. In refs. [5, 6] it was shown that the theory is parity-invariant if parity is taken to exchange the two SU(2)'s.



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However, new puzzles emerged. Ref. [6] studied the moduli space of the theory and found that it does not appear to match expectations for either two or three M2-branes. Additionally the spectrum of chiral primary operators had some "missing" operators that should have been present for a multiple M2-brane interpretation to be correct. In this work it was also noted that the level $k$ of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is a free discrete parameter which, at large values, causes the theory to become weakly coupled — but a finite set of M2-branes should not have any weakly coupled limit. Also, although ref. [4] found a result suggestive of compactification, it was not clear why going to the Coulomb branch should be related to a circle-compactified background.

In the present work we resolve some of the above puzzles. We conjecture that the Bagger-Lambert $A_4$ theory at level $k = 1$ describes the worldvolume dynamics of two M2-branes on the $\mathbb{Z}_2$ orbifold, defined by the uplift to M-theory of two D2-branes on an $O2^-$ orientifold. Equivalently, the level-one $A_4$ theory is the infrared fixed point of the SO(4) maximally supersymmetric Yang-Mills theory in 2+1 dimensions. With this interpretation, we argue that the spectrum of chiral operators is as expected. For general $k$, we argue that the moduli space is $(\mathbb{R}^8 \times \mathbb{R}^8)/D_{2k}$, where $D_{2k}$ is the dihedral group of order $4k$. We conjecture that this corresponds to two M2-branes on a $\mathbb{Z}_{2k}$ "M-fold", in which a geometrical action of $\mathbb{Z}_{2k}$ is combined with an action on the branes. Finally, we show that taking a large-$k$ limit at a point on moduli space where the branes are separated from the orbifold point, one recovers the worldvolume theory of D2-branes, as expected, since the orbifold locally becomes a cylinder.

We will work with the formulation of the $A_4$ theory in ref. [6]. The fields consist of two SU(2) gauge fields, having Chern-Simons actions with *opposite* signs, and a set of 8 scalar fields $X^I, I = 1, 2, \ldots, 8$ along with 8 fermions. All the matter fields transform as bi-fundamentals of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The action is:

$$\begin{aligned}\mathcal{L} = {} & \mathrm{tr}(-(D^\mu X^I)^\dagger D_\mu X^I + i\bar{\Psi}^\dagger \Gamma^\mu D_\mu \Psi) \\ & +\mathrm{tr}\bigg( -\frac{2}{3} i f \bar{\Psi}^\dagger \Gamma_{IJ}(X^I X^{J\dagger}\Psi + X^J \Psi^\dagger X^I + \Psi X^{I\dagger} X^J) - \frac{8}{3} f^2 X^{[I} X^{J\dagger} X^{K]} X^{K\dagger} X^J X^{I\dagger}\bigg) \\ & + \frac{1}{2f}\epsilon^{\mu\nu\lambda}\mathrm{tr}\left(A_\mu \partial_\nu A_\lambda + \frac{2}{3} i A_\mu A_\nu A_\lambda\right) - \frac{1}{2f}\epsilon^{\mu\nu\lambda}\mathrm{tr}\left(\hat{A}_\mu \partial_\nu \hat{A}_\lambda + \frac{2}{3} i \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda\right) . \end{aligned} \tag{1.1}$$

Here,

$$D_\mu X^I = \partial_\mu X^I + i A_\mu X^I - i X^I \hat{A}_\mu \,, \tag{1.2}$$

which is covariant under the action of the gauge transformations

$$X^I \to U X^I V^{-1}, \quad A_\mu \to U A_\mu U^{-1} + i\partial_\mu U U^{-1}, \quad \hat{A}_\mu \to V \hat{A}_\mu V^{-1} + i\partial_\mu V V^{-1} \,. \tag{1.3}$$

In the above, $f = 2\pi/k$ where $k$ is the (integer) level of the two Chern-Simons actions.[1] The supersymmetries under which the above action is invariant can be found in ref. [6].

Since our proposal involves orbifold 2-planes, let us briefly review some relevant facts. There are three types of orientifold 2-planes in type IIA string theory [8], denoted $O2^-$,

[1] The quantization of the Chern-Simons coefficient for a non-simply-connected gauge group, $G$, is a little subtle. See appendix A for details.

$\widetilde{O2}^+$ and $O2^+$, that give rise to gauge groups $\mathrm{SO}(2N), \mathrm{SO}(2N+1), \mathrm{Sp}(N)$ respectively when $N$ D2-branes are brought near them. All correspond to an inversion of 7 spatial directions transverse to the orientifold plane, and all can be uplifted to M-theory. The uplifted orientifold planes are really M-theory *orbifolds* rather than orientifolds, in the sense that they do not reverse the orientation of membranes or of the 3-form $C_{MNP}$.[2] This is because in IIA string theory, the $\mathbb{Z}_2$ action reverses the $B_{MN}$ field, but preserves the RR 3-form $C_{MNP}$. This implies an action on the M-circle as a reflection. After uplifting, the end result is that it preserves the 3-form of M-theory but reflects eight spatial directions including the M-circle. Due to their origin as orientifold planes, the M2-orbifold planes carry an M2-brane charge, which is $-\frac{1}{16}$ for the $O2^-$ case.

We can directly define the $O2^-$ plane in M-theory as the orbifold $R^8/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is diag $(-1,-1,-1,-1)$ on the four complex coordinates of $R^8$. With this $\mathbb{Z}_2$ action the supersymmetry near the plane is half-maximal and has 16 components just like the supersymmetry on M2 branes. This will turn out to be the case we understand best. For $k > 1$, the $\mathbb{Z}_2$ subgroup of $D_2$ associated with the inversion of the $\mathbb{R}^8$ is replaced with $\mathbb{Z}_{2k}$ in the definition of the moduli space. This suggests that the level $k$ M-fold combines a geometrical action of $\mathbb{Z}_{2k}$ with an action on the branes. While we will not be able to present a precise action of $\mathbb{Z}_{2k}$ satisfying all the requirements, we will discuss some possibilities in a subsequent section. Unlike $k = 1$, the general case is not likely to descend in a simple way to a type IIA orientifold since a $\mathbb{Z}_{2k}$ action with $k > 1$ will presumably mix the M-circle with another circle.[3]

In the rest of this note, we present evidence for our conjecture that the theory whose Lagrangian is given in eq. (1.1) describes two M2-branes at a $\mathbb{Z}_{2k}$ orbifold (with the action of $\mathbb{Z}_{2k}$ on the brane worldvolume fields appropriately defined).

When the original version of this paper was nearly complete, the paper [11] appeared, which has substantial overlap with our work. The original versions of our paper and of [11] differed in a few significant respects, but most of these differences have now been resolved in the revised versions. We thank David Tong and Neil Lambert for correspondence on these issues. Another very recent paper discussing multiple M2-branes is ref. [12].

## 2. Moduli space

The moduli space for the $A_4$ theory was studied in refs. [3, 6]. Here we will revisit this moduli space and argue that the complete moduli space at level $k$ is actually $(\mathbb{R}^8 \times \mathbb{R}^8)/D_{2k}$ once the gauge fields are taken into account. Here, $D_{2k}$ is the dihedral group, $\mathbb{Z}_2 \ltimes \mathbb{Z}_{2k}$ where the product is semidirect.

We begin with the action in eq. (1.1). As noted in [6], generic scalar configurations for which the potential vanishes correspond (up to gauge transformations) to diagonal matrices

[2] In contrast, orientifold 4-planes in type IIA lift to orientifold 5-planes in M-theory [9, 10].

[3] By compactifying a direction transverse to $R^8$ one can relate it to a type IIA orbifold, however in this case it becomes an orbifold 1-plane and carries the charge of fundamental strings rather than D2-branes.

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$X^I$, which we will parameterize by

$$X^I = \frac{1}{\sqrt{2}} \begin{pmatrix} z^I & 0 \\ 0 & \bar{z}^I \end{pmatrix} . \tag{2.1}$$

Within the space of these diagonal configurations, there is a residual $O(2)$ gauge symmetry, acting by simultaneous rotations on $z^I$ and by simultaneous complex conjugation. However, to describe the complete moduli space it will be important for us to take into account the gauge fields.

Generically, the diagonal configurations (2.1) break the gauge group down to U(1), and the remaining components of the gauge field become massive by the Higgs mechanism. Also, expanding the potential about such configurations shows that physical scalar fluctuations which take us away from a diagonal configuration are all massive.

We now write the classical action describing the dynamics of the light fields on the moduli space. To do this, it will be convenient to include both the preserved U(1) gauge field and the gauge field associated with the U(1) that rotates $z^I$. Together with the diagonal configuration (2.1), we take

$$A_\mu = \begin{pmatrix} a_\mu & 0 \\ 0 & -a_\mu \end{pmatrix}, \qquad \hat{A}_\mu = \begin{pmatrix} \hat{a}_\mu & 0 \\ 0 & -\hat{a}_\mu \end{pmatrix} \tag{2.2}$$

with the normalization chosen so that $a_\mu$ and $\hat{a}_\mu$ have gauge transformations

$$a_\mu \to a_\mu - \partial_\mu \theta, \qquad \hat{a}_\mu \to \hat{a}_\mu - \partial_\mu \hat{\theta} \tag{2.3}$$

where $\theta$ and $\hat{\theta}$ have period $2\pi$. This gives an action

$$S = \int d^3x \left( -\left|\partial_\mu z^I + i(a_\mu - \hat{a}_\mu) z^I\right|^2 + \frac{k}{2\pi} \epsilon^{\mu\nu\lambda} \left(a_\mu \partial_\nu a_\lambda - \hat{a}_\mu \partial_\nu \hat{a}_\lambda\right) \right) . \tag{2.4}$$

We further define

$$c_\mu = a_\mu + \hat{a}_\mu, \qquad b_\mu = a_\mu - \hat{a}_\mu \tag{2.5}$$

so that $c_\mu$ is the gauge field associated with the preserved U(1), and $b_\mu$ is associated with the U(1) that rotates $z^I$.

The resulting action is

$$S = \int d^3x \left( -\left|\partial_\mu z^I + i b_\mu z^I\right|^2 + \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} b_\mu f_{\nu\lambda} \right) , \tag{2.6}$$

where

$$f_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu . \tag{2.7}$$

The gauge transformations are

$$z^I \to e^{i(\theta - \hat{\theta})} z^I \qquad b_\mu \to b_\mu - \partial_\mu \theta + \partial_\mu \hat{\theta} \qquad c_\mu \to c_\mu - \partial_\mu \theta - \partial_\mu \hat{\theta} . \tag{2.8}$$

Note that the last term in the action is gauge invariant because of the Bianchi identity for $f$.



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To this action, we can add a Lagrange multiplier term

$$S_\sigma = \int d^3x \, \frac{1}{8\pi} \sigma(x) \epsilon^{\mu\nu\lambda} \partial_\mu f_{\nu\lambda} \tag{2.9}$$

and treat $f$ as an independent variable. The integral over $\sigma$ enforces the Bianchi identity. To be precise, we need $\sigma$ to be periodic with period $2\pi$. To see this, note that for monopole configurations we can have,[4]

$$\int d^3x \, \frac{1}{2} \epsilon^{\mu\nu\lambda} \partial_\mu f_{\nu\lambda} = \int_M dF = \int_{\partial M} F \in 4\pi\mathbb{Z} \, . \tag{2.10}$$

For this, it is essential to note that $f$ is the sum of field strengths for two gauge fields which are normalized conventionally (so the gauge transformation is the derivative of an angle without any numerical factors). So rather than a standard delta function, we want a periodic delta function that allows these monopole configurations. This is ensured by a $2\pi$ periodicity of $\sigma$.

Starting from the combined action

$$S = \int d^3x \left( -\left|\partial_\mu z^I + i b_\mu z^I\right|^2 + \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} b_\mu f_{\nu\lambda} + \frac{1}{8\pi} \sigma \epsilon^{\mu\nu\lambda} \partial_\mu f_{\nu\lambda} \right) \, , \tag{2.11}$$

the equation of motion for $f$ gives

$$b_\mu = \frac{1}{2k} \partial_\mu \sigma \, . \tag{2.12}$$

Using this, the full action reduces to

$$S = -\left|\partial_\mu z^I + \frac{i}{2k} z^I \partial_\mu \sigma\right|^2 \, . \tag{2.13}$$

The gauge invariance transformation on $b$ translates to a gauge invariance transformation on $\sigma$

$$z^I \to e^{i\alpha(x)} z^I \qquad\qquad \sigma \to \sigma - 2k\alpha(x) \, . \tag{2.14}$$

We can now fix our gauge to set $\sigma = 0$. After doing this, we still have residual gauge transformations

$$\alpha(x) = \frac{\pi n}{k} \, , \tag{2.15}$$

which leave $\sigma = 0$. The moduli space is therefore characterized by a set of eight complex numbers $z^I$, with gauge transformations that take

$$z^I \to e^{\pi i n/k} z^I \tag{2.16}$$

and

$$z^I \to \bar{z}^I \, . \tag{2.17}$$

[4] In this theory, the minimum monopole charge is double the one implied by the Dirac quantization condition. We justify this in appendix B.



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Here, the $\mathbb{Z}_2$ action and the $\mathbb{Z}_{2k}$ action don't commute with each other for $k > 1$, and the combined group is the dihedral group $D_{2k}$. We conclude that the moduli space for level $k$ is

$$(\mathbb{R}^8 \times \mathbb{R}^8)/D_{2k} \; . \tag{2.18}$$

For $k = 1$, this is just

$$(\mathbb{R}^8 \times \mathbb{R}^8)/(\mathbb{Z}_2 \times \mathbb{Z}_2) \; , \tag{2.19}$$

the moduli space of the superconformal theory that describes the infrared physics of maximally SUSY SO(4) Yang-Mills theory in 2+1 dimensions [8]. In contrast, the superconformal theory arising from SU(3) gauge theory should have moduli space[5]

$$(\mathbb{R}^8 \times \mathbb{R}^8)/S_3 \; . \tag{2.20}$$

For higher $k$, we conjecture that this theory describes the low-energy physics of two M2-branes in M-theory with a generalized orbifold action on $\mathbb{R}^8$. We expect that the geometrical action is $\mathbb{R}^8/\mathbb{Z}_{2k}$. However, the orbifold group must also act on the M2-brane fields, since the moduli space is not just $(\mathbb{R}^8/\mathbb{Z}_{2k})^2/\mathbb{Z}_2$.

The orbifold in question should preserve 16 supersymmetries and maximal SO(8) R-symmetry for all $k$ and have the desired action eq. (2.16) on moduli space. However, except for $k = 1, 2$ there appears to be no known orbifold with this property.[6] The most supersymmetric singularities of the form $\mathbb{R}^8/\mathbb{Z}_{2k}$ in M-theory are [13, 14]:

$$(z^1, z^2, z^3, z^4) \to (\omega z^1, \omega^{-1} z^2, \omega z^3, \omega^{-1} z^4) \tag{2.21}$$

where $\omega^{2k} = 1$. Perhaps surprisingly, this action preserves as many as 12 supersymmetries, or $\mathcal{N} = 6$ in 3 dimensions, and also gives rise to an R-symmetry $\mathrm{SU}(4)\times\mathrm{U}(1)$ [13, 14]. Even more intriguing, there are two exceptions to this rule — the cases with $k = 1, 2$. The former obviously preserves $\mathcal{N} = 8$, while the latter has also been claimed to do the same [13]. Though for general $k$ this orbifold does not appear to meet all the requirements, it is possible that it actually preserves more supersymmetry and R-symmetry than is apparent for reasons that we do not yet understand.[7]

For the present, since we do not have a precise formulation of the theory whose moduli space we have found, we simply think of it as the theory of 2 M2-branes on an "M-fold," and consider the $A_4$ theory at level $k$ to give a precise definition of the $\mathbb{Z}_{2k}$ "M-fold".

## 3. Chiral primary operators

In ref. [6], it was pointed out that it is impossible to construct operators in the $A_4$ theory which lie in tensor representations of SO(8) with an odd number of indices. This presented

[5] In general, the moduli space for gauge group $G$ with rank $n$ and Weyl group $\mathcal{W}$ is $\mathbb{R}^{8n}/\mathcal{W}$.

[6] We thank Nima Arkani-Hamed, Neil Lambert and David Tong for their comments on this issue.

[7] This could be an M-theory analogue of the mechanism in ref. [15] where a geometrical or supergravity analysis yields misleadingly low amounts of supersymmetry but additional stringy modes enhance the supersymmetry. In our system, the fact that $\mathbb{Z}_{2k}$ acts on the M2-branes may also be relevant.



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a puzzle for the interpretation of the $A_4$ theory as the worldvolume theory of a stack of M2-branes, since such theories are believed (at least for three or more M2-branes) to have a spectrum of chiral operators that includes these odd-indexed representations. We will now see that with our proposed interpretation, this is no longer a problem.

To see this most explicitly, let us focus on the case $k = 1$ and consider the UV gauge theory from which the superconformal field theory flows. For the SU(3) theory (or SU($N$) with $N > 2$), the scalar fields are seven Hermitian matrices, and we can construct operators

$$\mathrm{STr}(X^{i_1} \cdots X^{i_n}) - \mathrm{SO}(7) \text{ traces} \tag{3.1}$$

that should become a subset of the chiral primary operators in the infrared limit (the others are generated by the SO(8) rotations that are not manifest in the UV).[8] On the other hand, in the SO(4) gauge theory, the scalars are antisymmetric matrices, so the operators (3.1) vanish identically for odd numbers of indices. This strongly suggests that the chiral primary operators with odd numbers of SO(8) indices will not be present in the infrared theory either, so there is no conflict with identifying the IR limit of the SO(4) theory with the $k = 1$ $A_4$ theory.

Generally, we expect that superconformal field theories that have the same moduli space should also have the same spectrum of chiral operators [16], so perhaps the discussion in this section is somewhat redundant. However, it is interesting to understand explicitly why the odd-indexed representations do not show up in the SO(4) case.

## 4. Points on the moduli space

In this section, we discuss more explicitly the connection between points on the moduli space of the $A_4$ theory and configurations of M2-branes on an orbifold. We begin with the simplest case $k = 1$. Here, the conjecture is that the moduli space should coincide with the moduli space of two M2-branes on a $\mathbb{Z}_2$ orbifold. This arises in the strong coupling limit of type IIA string theory from a configuration of two D2-branes on an $O2^-$ orientifold.

To understand the properties of such an orbifold, let us begin by considering D2-branes at an $O2^-$ orientifold. The low-energy worldvolume theory for these is SO(4) maximally supersymmetric gauge theory. The scalars in this theory are antisymmetric $4 \times 4$ matrices, and configurations for which the scalar potential vanishes are gauge equivalent to

$$X^i = \begin{pmatrix} 0 & a^i & 0 & 0 \\ -a^i & 0 & 0 & 0 \\ 0 & 0 & 0 & b^i \\ 0 & 0 & -b^i & 0 \end{pmatrix} . \tag{4.1}$$

Residual gauge transformations preserving this form allow us to make the identifications

$$(a^i, b^i) \equiv (b^i, a^i) \equiv (-a^i, -b^i) \equiv (-b^i, -a^i) \,, \tag{4.2}$$

[8] Other trace structures give additional operators. For SU(2), such operators with an odd number of SO(7) indices do vanish identically, but for this case, the moduli space is only eight dimensional.

so the set of scalar field vevs with vanishing potential may be described by the space $(\mathbb{R}^7 \times \mathbb{R}^7)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

The full moduli space of the IR limit of the SO(4) gauge theory is $(R^8 \times R^8)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which we can describe by two vectors in $\mathbb{R}^8$, subject to the identifications

$$(A^I, B^I) \equiv (B^I, A^I) \equiv (-A^I, -B^I) \equiv (-B^I, -A^I)\ . \tag{4.3}$$

We can interpret $A$ and $B$ as the locations of the two M2-branes. However, note that $(A^I, B^I)$ and $(A^I, -B^I)$ are inequivalent, so the moduli space is not just a product of two $\mathbb{R}^8/\mathbb{Z}_2$'s divided by the symmetric group, as one might naively expect. In this characterization, a special role is played by configurations $(A^I, A^I)$ where the branes are coincident. These are invariant under the transformations

$$(A^I, B^I) \equiv (B^I, A^I) \tag{4.4}$$

and so lie at special points of the moduli space.

Now, going back to the $A_4$ theory for $k = 1$, we had derived the moduli space as the space of complex vectors $z^I$ up to gauge transformations

$$z^I \to -z^I \qquad\qquad z^I \to \bar{z}^I\ . \tag{4.5}$$

The first of these has no nontrivial fixed points, while the second has a fixed point for real vectors. Thus, for our choice of gauge, it is natural to make the associations

$$\begin{aligned} Re(z^I) &= A^I + B^I \\ Im(z^I) &= A^I - B^I \end{aligned} \tag{4.6}$$

so that the fixed points of complex conjugation (equivalently, the special points on the moduli space preserving SU(2)) are identified with coincident branes. It may seem puzzling at first that there seem to be more configurations than those with $z^I$ real that preserve SU(2) symmetry, namely any set of $z^I$ which lie in a line on the complex plane. However, for these configurations, we do not have a free abelian gauge field that can be dualized to a scalar, so these are all gauge-equivalent to the configurations with $z^I$ real.

So far the discussion has dealt with $k = 1$. It would be nice to carry out a similar analysis for higher $k$, in particular to find the precise relation between our coordinates $z^I$ on the moduli space and the positions of the branes.

For $k > 1$, we can also offer a rather heuristic geometrical explanation for the origin of $D_{2k}$ as follows (we expect this explanation could be made more precise with a better understanding of the "M-fold"). Suppose we bring two M2-branes to a $\mathbb{Z}_{2k}$ orbifold. It is plausible that the result is, to start with, a theory on $2k$ copies of the original branes, namely an $\mathrm{SU}(2)^{2k}$ quiver gauge theory. The quiver diagram is a $2k$-gon with the gauge fields at the vertices. The form of the action eq. (1.1) is consistent with the presence of $2k$ SU(2)'s, except that at the origin of moduli space the orbifold plane causes $k$ of these SU(2)'s to get identified with each other, so that their action is $k$ times the action of a single (level-1) SU(2) Chern-Simons gauge field. Likewise, the other $k$ SU(2)'s get identified and



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their action is $k$ times that of another level-1 SU(2) Chern-Simons gauge field, appearing in the action with a negative sign. Given the quiver interpretation, the associated symmetry group should be the set of all discrete transformations that map the quiver to itself. This includes cyclic rotations as well as reflections along any axis joining opposite vertices. By definition, this is the group of symmetries of a $2k$-gon, namely $D_{2k}$.

## 5. Large $k$ and compactification

In this section, we will see that the findings in ref. [4] fit very naturally with our interpretation. In that paper it was found that expanding the $A_4$ action about a special point on the moduli space where SU(2) gauge symmetry is preserved gives an action which is at leading order the maximally supersymmetric U(2) Yang-Mills theory. The extra U(1) comes from dualizing the scalar field that corresponds to multiplying all vevs by a constant. This is not really a free scalar but is approximately free at large distances from the orbifold, corresponding to the fact that the theory on two M2-branes effectively has a centre-of-mass mode when the branes are far away from the orbifold plane.

The procedure of [4] gives the Yang-Mills action plus an infinite series of higher dimension operators. While the latter can be decoupled in the limit $g_{\rm YM} \to \infty$, the Yang-Mills action simultaneously becomes strongly coupled in this limit. So there is no limit where one really has finitely coupled D2-branes. However, the the analysis of [4] was for level $k = 1$. Repeating it for general $k$, we find the following. By rescaling $X \to \sqrt{k}X, \Psi \to \sqrt{k}\Psi$, we easily see that the action eq. (1.1) acquires an overall multiplicative factor of $k$. Denoting this scaled action for the level-$k$ theory as $\mathcal{L}^{(k)}$, we have

$$\mathcal{L}^{(k)} = k\,\mathcal{L}^{(k=1)}\,. \tag{5.1}$$

Now in ref. [4] the action $\mathcal{L}^{(k=1)}$ was examined in the presence of a large vev $\langle X^{\phi(8)}\rangle = v$ (there, this vev was called $R$ and later $g_{\rm YM}$). It was shown (see eq. (3.23) of that reference) that

$$\mathcal{L}^{(k=1)} = \frac{1}{v^2}\mathcal{L}_0 + \frac{1}{v^3}\mathcal{L}_1 + \mathcal{O}\left(\frac{1}{v^4}\right) \tag{5.2}$$

where $\mathcal{L}_0$ is the action for an $N = 8$ SU(2) Yang-Mills theory.

For the Lagrangian $\mathcal{L}^{(k)}$, we must define the Yang-Mills coupling by

$$g_{\rm YM}^2 = \frac{v^2}{k}\,. \tag{5.3}$$

Taking the limit $k \to \infty, v \to \infty$ with $g_{\rm YM}$ fixed, we see that the Yang-Mills part of the action has a finite coupling $g_{\rm YM}$. However, successive terms scale to zero in this limit. Therefore in the limit we obtain precisely the D2-brane worldvolume theory with a tunable *finite* gauge coupling $g_{\rm YM}$, and no higher dimension operators.

With our interpretation of the theory, this observation is exactly what we would expect. We have argued that points on the moduli space preserving SU(2) correspond to taking two coincident M2-branes away from an orbifold fixed point. While a precise definition of the orbifold action is lacking, for this discussion it is sufficient to assume it leads to an



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opening angle that shrinks like $1/k$ as is the case for standard orbifolds. Now, in the limit where $k \to \infty$, the opening angle of the orbifold goes to zero, so at some point sufficiently far out on the moduli space, the local geometry approaches that of a cylinder $\mathbb{R}^7 \times S^1$. The scaling limit below eq. (5.3) precisely takes the two M2-branes out into this cylindrical space, where they should behave like two D2-branes in type IIA string theory. So we expect a finitely coupled U(2) Yang-Mills theory — and that is exactly what we find.

## 6. Discussion

In this paper, we have found that the moduli space for the Bagger-Lambert $A_4$ theory at level $k$ is $(\mathbb{R}^8 \times \mathbb{R}^8)/D_{2k}$, where $D_{2k}$ is the dihedral group of order $4k$. Our interpretation is that the theory describes M2-branes on a $\mathbb{Z}_{2k}$ "M-fold," a generalization of the $\mathbb{Z}_2$ case defined by the uplift of the $O2^-$ orientifold in string theory.

We feel compelled to mention that the superconformal theories defined as the IR fixed point of U(2), SO(4), SU(3), SO(5), and $G_2$ all have moduli spaces of the form

$$(\mathbb{R}^8 \times \mathbb{R}^8)/\mathcal{W} \tag{6.1}$$

where $\mathcal{W}$ is respectively $D_1$, $D_2$, $D_3$, $D_4$, and $D_6$. Within our interpretation, only the identification of the level $k = 1$ theory with $D_2$ seems natural, however, it may be that the level $k = 2$ and $k = 3$ cases happen to coincide with the infrared limit of SO(5) and $G_2$ maximally supersymmetric gauge theory respectively.[9] This must be true unless there exist pairs of distinct SO(8) superconformal field theories with the same moduli space.

The discussion in the limit of large-order $\mathbb{Z}_{2k}$ orbifolds bears a strong resemblance to the deconstruction approach to M5-branes discussed in ref. [17]. In section IIIB of that paper, a limit is taken where the order of the orbifold grows large and simultaneously the D-branes are moved far away from the orbifold so that effectively they end up propagating on a cylinder. It would be interesting to explore whether the corresponding limit in our paper is related to deconstruction and M5-branes.

## A. Chern-Simons level quantization

The purpose of this appendix is to review the quantization of the Chern-Simons level for the non-simply connected gauge groups, SO($n$). None of the results are original but, particularly for the spin Chern-Simons case, they are not as well-known as they should be. We would like to thank Dan Freed for guiding us through the computation below.

As shown by Dijkgraaf and Witten [18], the level of a Chern-Simons theory with gauge group, $G$, is specified by an element $a \in H^4(BG)$. If $H^4(BG)$ is one-dimensional, then $a$ is an integer multiple of the generator, and the choice of Chern-Simons action comes down to specifying that integer. When $G$ is not simply connected, it is convenient to write the

[9] As we mentioned in section 2, there is a plausible maximally supersymmetric orbifold [13] for precisely $k = 2$, suggesting a distinctive role for this case along with $k = 1$. The $k = 2$ orbifold is related to the strong coupling limit of D2-branes at an $O2^+$ orientifold. A more detailed discussion of this case may be found in [11].

normalization of the Chern-Simons action for $G$ *relative* to that of the Chern-Simons action for the simply connected covering group, $\tilde{G}$. That is, the homomorphism $\tilde{G} \to G$ induces a map $H^4(BG) \to H^4(B\tilde{G})$, and we might wish to note that the generator of $H^4(BG)$ maps to some *multiple* of the generator of $H^4(B\tilde{G})$.

On a spin-manifold (which we certainly have, in our case, as we are interested in supersymmetric theories), one can define a refined version of Chern-Simons theory, called spin Chern-Simons [19]. The precise definition of the action is slightly more subtle (it involves the choice of a spin structure on the 3-manifold), but the *variation* of the action, as one varies the gauge connection is the same as for conventional Chern-Simons. The only difference, from our perspective, is that the quantization condition on the level is somewhat relaxed.

The level for spin Chern-Simons is specified by an element, $a \in E^4(BG)$, of what Dan Freed calls [20] $E$-cohomology, which combines information from the integer cohomology with some mod-2 information. In particular, for any space, $X$, there is a long exact sequence

$$\cdots \to H^n(X) \to E^n(X) \to H^{n-2}(X, \mathbb{Z}/2) \xrightarrow{\beta\circ Sq^2} H^{n+1}(X) \to E^{n+1}(X) \to \dots \tag{A.1}$$

where the connecting homomorphism is the integer Bockstein, composed with the second Steenrod square.

Let's apply this to the classifying spaces for $G = \mathrm{SO}(n)$, $\tilde{G} = Spin(n)$. For $n \geq 5$, $n = 3$, $H^4(BSO(n)) = \mathbb{Z}$, with generator $p_1$. $H^4(BSO(4)) = \mathbb{Z} \oplus \mathbb{Z}$, with generators $p_1$ and $e$. $H^2(BSO(n), \mathbb{Z}/2) = \mathbb{Z}/2$, with generator $w_2$. The above long exact sequence gives rise to short exact sequences

$$\begin{array}{ccccccc} 0 \to & H^4(BSpin(n)) & \xrightarrow{\simeq} & E^4(BSpin(n)) & \to 0 & & \\ & \alpha \uparrow & & \gamma \uparrow & & & \\ 0 \to & H^4(BSO(n)) & \xrightarrow{\beta} & E^4(BSO(n)) & \to & H^2(BSO(n), \mathbb{Z}/2) & \to 0 \end{array} \tag{A.2}$$

For $n \geq 5$, the map $\alpha$ is multiplication by 2 ($p_1$ for a $Spin(n)$ bundle is always even), as is the map $\beta$. Hence the map $\gamma$ is an isomorphism. Thus, for $\mathrm{SO}(n)$, $n \geq 5$, the Chern-Simons coefficient $k$ must be even, while the spin Chern-Simons coefficient can be any integer.

For $n = 3$ the map $\alpha$ is multiplication by 4 (see, e.g., equation (4.11) of [18]). $\beta$ is still multiplication by 2, hence $\gamma$ is multiplication by 2. Thus, for ordinary SO(3) Chern-Simons, $k \in 4\mathbb{Z}$, whereas for SO(3) spin Chern-Simons, $k \in 2\mathbb{Z}$.

For SO(4), the case of actual interest in this paper, the map $\alpha = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ has index-2. The map $\beta$ also has index-2. Hence $\gamma$ is an isomorphism. Thus, for ordinary SO(4) Chern Simons, the two SU(2) levels satisfy $k_{L,R} \in \mathbb{Z}$, with $k_L - k_R \in 2\mathbb{Z}$. For SO(4) spin Chern-Simons, $k_{L,R}$ can be any integers.

As it turns, for the Bagger-Lambert $A_4$, we are interested in $k_L = k$, $k_R = -k$, so the carry-away from this analysis is that any integer value of $k$ is allowed, and there is no distinction between the ordinary and spin Chern-Simons cases.

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## B. Monopole charge quantization

In this appendix, we briefly explain why the quantization (2.10) of the monopole charge in this theory is such that the minimum charge is twice the one implied by Dirac's quantization condition. It is well known that for 't Hooft-Polyakov monopoles, the minimum charge is actually twice the Dirac value [21]. In that case, all fields transform in the adjoint representation of the gauge group SU(2), so effectively the gauge group is SO(3).

In the theory we are considering, the gauge field $c_\mu$ corresponding to the unbroken U(1) at generic points on the moduli space sits inside the diagonal $\mathrm{SO}(3) \in (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathbb{Z}_2$, and all the matter fields transform in the adjoint of this SO(3), so the situation sounds similar to the 't Hooft-Polyakov case. However, while monopole configurations in the 't Hooft-Polyakov case are classified by $\pi_2(\mathrm{SO}(3)/\mathrm{SO}(2))$,[10] monopole configurations in the BL theory should be classified by elements of $\pi_2(\mathrm{SO}(4)/\mathrm{SO}(2))$. The embedding of the diagonal SO(3) in SO(4) induces a natural map

$$\pi_2(\mathrm{SO}(3)/\mathrm{SO}(2)) \to \pi_2(\mathrm{SO}(4)/\mathrm{SO}(2)) \sim \mathbb{Z} \to \mathbb{Z} \tag{B.1}$$

but it is not obvious that this is an isomorphism. In particular, if the generator of $\pi_2(\mathrm{SO}(3)/\mathrm{SO}(2))$ mapped to the square of the generator of $\pi_2(\mathrm{SO}(4)/\mathrm{SO}(2))$, the BL theory would contain monopoles with the minimal Dirac charge. It turns out that there can be no such configurations, since the map (B.1) is onto, as we will now show using the following theorem [22].

**Theorem:** if $p : Y \to B$ is a fibration and if $y_0 \in Y$, $b_0 = p(y_0)$, and $F = p^{-1}(b_0)$, then taking $y_0$ as the base point of $Y$ and of $F$ and $b_0$ as the base point of $B$, we have the exact sequence:

$$\cdots \to \pi_n(F) \to \pi_n(Y) \to \pi_n(B) \to \pi_{n-1}(F) \to \dots$$

For our purposes, we take

$$\begin{aligned} Y &= \mathrm{SO}(4)/\mathrm{SO}(2) = (\mathrm{SU}(2)\times\mathrm{SU}(2))/\mathrm{U}(1) \\ F &= \mathrm{SO}(3)/\mathrm{SO}(2) = \mathrm{SU}(2)/\mathrm{U}(1) \\ B &= \mathrm{SU}(2)\ . \end{aligned}$$

We can represent $Y$ by pairs $(U, V)$ of SU(2) matrices where

$$(U, V) \simeq (Ue^{i\theta\sigma_3}, Ve^{i\theta\sigma_3})\ .$$

Take the fibration map $p$ to be $(U, V) \to UV^{-1}$. We can take $y_0 = (1, 1)$ so that $F$ is the subgroup of $Y$ such that $U = V$, which is $\mathrm{SU}(2)/\mathrm{U}(1) = \mathrm{SO}(3)/\mathrm{SO}(2)$. Now, a particular part of the exact sequence is

$$\cdots \to \pi_2(\mathrm{SO}(3)/\mathrm{SO}(2)) \xrightarrow{a} \pi_2(\mathrm{SO}(4)/\mathrm{SO}(2)) \to \pi_2(\mathrm{SU}(2)) \to \dots$$

Since $\pi_2(\mathrm{SU}(2)) = 0$, exactness implies that the map $a$ is onto, as we wanted to show.

[10] In general, monopole solutions are classified by $\pi_2(G/H)$ where $G$ is the gauge group and $H$ is the unbroken subgroup.



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